PERGAMON

International Journal of Heat and Mass Transfer 42 (1999) 1513–1520

International Journal of HEAT and MASS TRANSFER

A variance propagation algorithm for the computation of heat conduction under stochastic conditions

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Received 13 March 1997; in final form 20 August 1997

Abstract

In this article a variance propagation algorithm is derived for the computation of heat conduction problems with parameters which fluctuate stochastically in time. The method is based on the finite element discretization of the Fourier equation. The algorithm involves the numerical solution of systems of Sylvester and Lyapunov equations and has been implemented on top of an existing finite element heat conduction package. Based on some illustrative examples, it is shown that the method compares favourably to the Monte Carlo method in terms of accuracy and computer time.

Nomenclature

a_i coefficients of AR(m) process
A state space companion matrix
B auxiliary vector in state space form of autoregressive random process
C finite element capacitance matrix
E mean value operator
$f(x,t)$ probability density function of $x(t)$
f finite element thermal load vector
G heat conduction domain
h surface heat transfer coefficient [W/m^2 °C]
I identity matrix
k thermal conductivity [W/m °C]
K finite element conductance matrix
L length of slab [m]
n_{nod} number of nodes
n outward normal to convection surface
0 null matrix
Q heat generation rate [W/m^3]
t time [s]
T temperature [°C]
T_0 initial temperature [°C]
T_∞ ambient temperature [°C]
u nodal temperature vector
$\mathbf{V}_{\mathbf{a,b}}$ covariance matrix of random vectors **a** and **b**
w Gaussian white noise process
x arbitrary random parameter
x state vector in state space form of autoregressive process
z position vector.

Greek symbols
∂G boundary surface
δ Dirac delta
Δt time step [s]
ρc volumetric heat capacity [J/m^3 °C]
σ standard deviation.

1. Introduction

For the numerical solution of heat conduction problems it is usually assumed that all parameters are known. In reality, the product properties (thermophysical properties) as well as the external parameters (initial and ambient temperature, surface heat transfer coefficient) may change randomly both as a function of space and time coordinates. For example, the chemical composition and physical structure of many agricultural materials are

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0017–9310/98/$ - see front matter
PII: S0017–9310(97)00279–2

very heterogeneous and are a function of several factors such as the harvest time and weather conditions, and the origin of the product. The thermophysical properties are affected by this heterogeneity and can vary inside the product as a function of the space coordinates. These properties are essentially deterministic, and if a method would be available to measure them with sufficient spatial resolution and accuracy they could be used as such in the numerical solution of the heat conduction problem. However, because of experimental limitations this is usually inpractical, and often it is more convenient to consider them as multidimensional random quantities. Other parameters, such as the temperature of the convective fluid flowing around the conducting solid, are intrinsically stochastic and may change in an unpredictable way during the heat transfer process. For example, the temperature inside a refrigerated room may fluctuate as a function of time as a consequence of unpredictable openings of refrigerator or oven doors, actions of the temperature control system, ambient conditions outside the refrigerated room, changes of solar radiation flux. As a consequence, the temperature distribution inside the heated object is also random and can only be specified meaningfully by means of statistical characteristics such as its mean value, variance and probability density function.

A straightforward statistical approach to the solution of random heat conduction problems is the Monte Carlo method. In this method a (pseudo)random sample of the stochastic parameters is generated by the computer and the corresponding heat transfer problem is numerically solved. The solution is stored and the process is repeated for a large number of times. In the end, the statistical characteristics of the transient temperature field are estimated using classical inference techniques. The considerable number of repetitive simulations required to obtain an acceptable level of accuracy is a major drawback of the Monte Carlo method, particularly when the heat transfer problem must be solved by means of a finite difference or finite element method. Further, the stochastic parameter set must be completely specified in the probabilistic sense, including probability density functions.

Several alternative methods have been proposed in the literature (for an overview, see ref. [1]). Of particular interest are those based on finite difference and finite element methods, as they can be applied where no analytical solutions can be obtained. Finite element perturbation methods have been discussed by several authors [2, 3, 4] to compute the mean and the variance of the temperature at arbitrary space-time coordinates in the heated objects for parameters of the random variable or random field type. Tasaka [5, 6] calculated statistical moments up to the second order of linear one-dimensional heat conduction problems with stochastic initial and boundary conditions using analytical and numerical finite element/finite difference based techniques. Nicolaï and De Baerdemaeker [7] derived a variance propagation algorithm for the solution of heat conduction problems with stochastic initial and boundary conditions. The algorithm involved the numerical solution of a Lyapunov matrix differential equation. Madera [1] suggested a method to calculate mean values, correlations and variances of non-stationary temperature fields in three-dimensional bodies of complicated shape and with random coefficients and stochastic initial and boundary conditions. The method involves the stochastic finite element and finite difference discretization of the random heat conduction equation and the solution of Volterra stochastic integral equations. The moments of the temperature vector are obtained in the form of converging matrix series.

In this article the variance propagation algorithm described in ref. [7] is extended to stochastic heat conduction problems with autoregressive parameters of arbitrary order. No spatial randomness will be considered.

2. Stochastic heat conduction model

It is assumed that the temperature field $T(\mathbf{z}, t)$, with $\mathbf{z}$ and t the space and time coordinate, is described by the linear heat conduction equation defined over a spatial domain G with boundary ∂G

$$k\nabla^2 T + Q = \rho c \frac{\partial T}{\partial t} \tag{1}$$

subjected to the initial and convection boundary condition

$$T(\mathbf{z}, t) = T_0(\mathbf{z}) \quad \text{at } t = t_0 \tag{2}$$

$$k\frac{\partial}{\partial n} T(\mathbf{z}, t) = h(t)[T_\infty(t) - T(\mathbf{z}, t)] \quad \text{on } \partial G \tag{3}$$

It is assumed that all parameters are stationary random processes. The mean $\bar{x}$ and covariance $V_{x,x}$ of a stationary process x with probability density function $f(x, t)$ are defined by

$$\bar{x} = E(x) \triangleq \int_{-\infty}^{\infty} x f(x, t)\,\mathrm{d}x$$

$$V_{x,x}(\Delta t) = E\{[x(t) - \bar{x}][x(t + \Delta t) - \bar{x})]\}$$

A Gaussian stationary white noise process w with covariance

$$V_{w,w}(\Delta t) = \sigma_w^2 \delta(\Delta t) \tag{4}$$

where δ is the Dirac delta, can be used to describe very rapid unpredictable fluctuations. Sample values of w are uncorrelated no matter how close together in time they are. However, white noise does not exist in reality as it has an infinite energy content. Autoregressive processes

provide a tool to incorporate fluctuations which change more smoothly as a function of time. An autoregressive random process of order m is defined by the following stochastic differential equation

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}x(t)+a_1\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}x(t)+\cdots+a_m x(t)=w(t) \tag{5}$$

where $a_1, a_2, \ldots, a_m$ are constants, $m \geqslant 1$, and $w(t)$ is a stationary Gaussian white noise process with mean $\bar{w}=\alpha_m\bar{\mathbf{x}}$. The time scale of the fluctuations is dependent on the coefficients $a_1, \ldots, a_m$, and their high frequency content decreases with increasing order m.

The (Gaussian) random variable initial condition corresponding to the stochastic differential equation (5) is defined as

$$E[x(t_0)]=\bar{x} \tag{6}$$

$$E[x(t_0-x)]^2=\sigma_x^2 \tag{7}$$

Note that a random variable parameter x can be modeled as a trivial case of an AR(1) process:

$$\frac{\mathrm{d}}{\mathrm{d}t}x=0 \tag{8}$$

AR(m) processes are a special case of the class of physically realizable stochastic processes which comprise most of the random processes seen in practice [8].

It is convenient to write the autoregressive process (5) in the following state space form [9]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}w(t) \tag{9}$$

where

$$\mathbf{x}=\begin{bmatrix} x \\ \mathrm{d}x/\mathrm{d}t \\ \vdots \\ \mathrm{d}^{m-2}x/\mathrm{d}t^{m-2} \\ \mathrm{d}^{m-1}x/\mathrm{d}t^{m-1} \end{bmatrix} \quad \mathbf{B}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ -a_m & -a_{m-1} & \ldots & -a_1 \end{bmatrix}$$

The vector $\mathbf{x}$ is called the state vector, the matrix $\mathbf{A}$ the companion matrix and the vector $\mathbf{B}$ is an auxiliary vector. In this article it will be assumed that T_∞, h and Q are autoregressive processes of order m_{T_∞}, m_h, and m_Q, respectively, as defined by the following state space equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_{T_\infty}(t)=\mathbf{A}_{T_\infty}\mathbf{x}_{T_\infty}(t)+\mathbf{B}_{T_\infty}w_{T_\infty}(t) \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_h(t)=\mathbf{A}_h\mathbf{x}_h(t)+\mathbf{B}_h w_h(t) \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_Q(t)=\mathbf{A}_Q\mathbf{x}_Q(t)+\mathbf{B}_Q w_Q(t) \tag{12}$$

with w_{T_∞}, w_h, w_Q white noise processes of in general different covariance. Observe that, whereas the time dependency of T_∞ and Q is obvious, h is traditionally assumed to be dependent of the temperature only. However, according to Newton's law of cooling h is defined as the proportionality constant between local heat flux and the temperature difference between the surface of the conductive medium and the fluid. The values of the local heat flux and temperature difference between the boundary of the conductive medium and the fluid can be measured using heat flux sensors and temperature probes and generally change in time. Consequently, according to the above definition h can also vary as a function of time. Although the physical significance of an instantaneous surface heat transfer coefficient can be questioned, in this article a pragmatic approach will be followed and h will be considered as a parameter which depends on time. The presence of stochastic disturbances can be expected in turbulent conditions, where the velocity of the fluid, and, hence, the surface heat transfer coefficient, is known to fluctuate randomly around a mean value.

As the thermophysical properties k and ρc usually do not change as a function of time, they are modeled as random variables by means of the following trivial differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}k=\frac{\mathrm{d}}{\mathrm{d}t}\rho c=0 \tag{13}$$

with appropriate initial conditions. Obviously, T_0 is modeled as a random variable as well.

3. Variance propagation algorithm

Realistic heat conduction problems often involve complicated geometrical shapes, non-uniform initial temperature distributions and time-varying surface heat transfer coefficient or ambient temperatures. Usually no analytical solution is available, and the Fourier equation must be solved using an appropriate numerical technique. In this article the finite element method has been applied.

Galerkin finite element discretization of equation (1) subject to equations (2) and (3) over a mesh of n_{nod} nodes yields the following differential system [10]

$$\mathbf{C}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}+\mathbf{K}\mathbf{u}=\mathbf{f} \tag{14}$$

with $\mathbf{u}=[u_1\, u_2 \ldots u_{n_{\mathrm{nod}}}]^T$ the overall nodal temperature vector, $\mathbf{C}$ the capacitance matrix and $\mathbf{K}$ the conductance matrix, both $n_{\mathrm{nod}} \times n_{\mathrm{nod}}$ matrices, and $\mathbf{f}$ a $n_{\mathrm{nod}} \times 1$ vector. $\mathbf{K}$ is a function of k and h; $\mathbf{C}$ of ρc, and $\mathbf{f}$ of h, Q and T_∞.

Equations (14) and (10)–(13) can be stacked conveniently in the following global system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{g}[\mathbf{x}(t), t] + \mathbf{h}[\mathbf{x}(t), t]\mathbf{w}(t) \tag{15}$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_{T_\infty} \\ \mathbf{x}_h \\ \mathbf{x}_Q \\ k \\ \rho c \end{bmatrix} \tag{16}$$

$$\mathbf{g} = \begin{bmatrix} \mathbf{C}^{-1}(-\mathbf{K}\mathbf{u} + \mathbf{f}) \\ \mathbf{A}_{T_\infty}\mathbf{x}_{T_\infty} + \mathbf{B}_{T_\infty}\bar{w}_{T_\infty} \\ \mathbf{A}_h\mathbf{x}_h + \mathbf{B}_h\bar{w}_h \\ \mathbf{A}_Q\mathbf{x}_Q + \mathbf{B}_Q\bar{w}_Q \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

$$\mathbf{h} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{T_\infty} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_Q \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{18}$$

$$\mathbf{w} = \begin{bmatrix} w_{T_\infty} - \bar{w}_{T_\infty} \\ w_h - \bar{w}_h \\ w_Q - \bar{w}_Q \end{bmatrix} \tag{19}$$

with $\mathbf{0}$ null vectors of appropriate dimension, and

$$\mathbf{V}_{\mathbf{w},\mathbf{w}}(\Delta t) \triangleq E[\mathbf{w}(t)\mathbf{w}^T(t+\Delta t)] = \delta(\tau)\begin{bmatrix} \sigma^2_{w,T_\infty} & 0 & 0 \\ 0 & \delta^2_{w,h} & 0 \\ 0 & 0 & \delta^2_{w,Q} \end{bmatrix} \tag{20}$$

It can be shown [11] that a first order approximation of the mean value $\bar{\mathbf{x}} \triangleq E(\mathbf{x})$ and the covariance matrix $\mathbf{V}_{\mathbf{x},\mathbf{x}} \triangleq E[(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}}^T)]$ of $\mathbf{x}$ can be calculated at an arbitrary time from the following system

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathbf{x}}(t) = \mathbf{g}[\bar{\mathbf{x}}(t), t] \tag{21}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_{\mathbf{x},\mathbf{x}}(t) = {} & \frac{\partial \mathbf{g}[\bar{\mathbf{x}}(t), t]}{\partial \bar{\mathbf{x}}}\mathbf{V}_{\mathbf{x},\mathbf{x}}(t) \\ & + \mathbf{V}_{\mathbf{x},\mathbf{x}}(t)\left\{\frac{\partial \mathbf{g}[\bar{\mathbf{x}}(t), t]}{\partial \bar{\mathbf{x}}}\right\}^T \\ & + \mathbf{h}[\bar{\mathbf{x}}(t), t]\mathbf{V}_{\mathbf{w},\mathbf{w}}(t)\mathbf{h}^T[\bar{\mathbf{x}}(t), t] \end{aligned} \tag{22}$$

After substitution of equations (16)–(19) in equations (21) and (22) and subsequent rearrangement, the following system is obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathbf{u}} = \bar{\mathbf{C}}^{-1}(-\bar{\mathbf{K}}\bar{\mathbf{u}} + \bar{\mathbf{f}}) \tag{23}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_{\mathbf{u},\mathbf{u}} = {} & \bar{\mathbf{C}}^{-1}\left[-\bar{\mathbf{K}}\bar{\mathbf{u}}\mathbf{V}_{\mathbf{u},\mathbf{u}} - \frac{\partial \mathbf{K}}{\partial k}\bar{\mathbf{u}}\mathbf{V}^T_{\mathbf{u},k}\right. \\ & - \frac{\partial \mathbf{C}}{\partial \rho c}\frac{\mathrm{d}\bar{\mathbf{u}}}{\mathrm{d}t}\mathbf{V}^T_{\mathbf{u},\rho c} + \frac{\partial \mathbf{f}}{\partial T_\infty}\mathbf{V}^T_{\mathbf{u},T_\infty} \\ & \left. + \left(\frac{\partial \mathbf{f}}{\partial h} - \frac{\partial \mathbf{K}}{\partial h}\bar{\mathbf{u}}\right)\mathbf{V}^T_{\mathbf{u},h} + \frac{\partial \mathbf{f}}{\partial Q}\mathbf{V}^T_{\mathbf{u},Q}\right] \\ & + \left[-\bar{\mathbf{K}}\bar{\mathbf{u}}V_{\mathbf{u},\mathbf{u}} - \frac{\partial \mathbf{K}}{\partial k}\bar{\mathbf{u}}\mathbf{V}^T_{\mathbf{u},k} - \frac{\partial \mathbf{C}}{\partial \rho c}\frac{\mathrm{d}\bar{\mathbf{u}}}{\mathrm{d}t}\mathbf{V}^T_{\mathbf{u},\rho c}\right. \\ & \left. + \frac{\partial \mathbf{f}}{\partial T_\infty}\mathbf{V}^T_{\mathbf{u},T_\infty} + \left(\frac{\partial \mathbf{f}}{\partial h} - \frac{\partial \mathbf{K}}{\partial h}\bar{\mathbf{u}}\right)\mathbf{V}^T_{\mathbf{u},h} + \frac{\partial \mathbf{f}}{\partial Q}\mathbf{V}^T_{\mathbf{u},Q}\right]^T \bar{\mathbf{C}}^{-T} \end{aligned} \tag{24}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_{\mathbf{u},\mathbf{x}_{T_\infty}} = \bar{\mathbf{C}}^{-1}\left(-\bar{\mathbf{K}}\mathbf{V}_{\mathbf{u},\mathbf{x}_{T_\infty}} + \frac{\partial \mathbf{f}}{\partial T_\infty}\mathbf{V}_{T_\infty,\mathbf{x}_{T_\infty}}\right) + \mathbf{V}_{\mathbf{u},\mathbf{x}_{T_\infty}}\mathbf{A}^T_{T_\infty} \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_{\mathbf{u},\mathbf{x}_h} = \bar{\mathbf{C}}^{-1}\left[-\bar{\mathbf{K}}\mathbf{V}_{\mathbf{u},\mathbf{x}_h} + \left(\frac{\partial \mathbf{f}}{\partial h} - \frac{\partial \mathbf{K}}{\partial h}\bar{\mathbf{u}}\right)\mathbf{V}_{h,\mathbf{x}_h}\right] + \mathbf{V}_{\mathbf{u},\mathbf{x}_h}\mathbf{A}^T_h \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_{\mathbf{u},\mathbf{x}_Q} = \bar{\mathbf{C}}^{-1}\left(-\bar{\mathbf{K}}\mathbf{V}_{\mathbf{u},\mathbf{x}_Q} + \frac{\partial \mathbf{f}}{\partial Q}\mathbf{V}_{Q,\mathbf{x}_Q}\right) + \mathbf{V}_{\bar{\mathbf{u}},\mathbf{x}_Q}\mathbf{A}^T_Q \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_{\mathbf{u},k} = \bar{\mathbf{C}}^{-1}\left(-\bar{\mathbf{K}}\mathbf{V}_{\mathbf{u},k} - \sigma^2_k\frac{\partial \mathbf{K}}{\partial k}\bar{\mathbf{u}}\right) \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_{\mathbf{u},\rho c} = \bar{\mathbf{C}}^{-1}\left(-\bar{\mathbf{K}}\mathbf{V}_{\mathbf{u},\rho c} - \sigma^2_{\rho c}\frac{\partial \mathbf{C}}{\partial \rho \mathbf{c}}\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathbf{u}}\right) \tag{29}$$

where the notation $\bar{\mathbf{C}}^{-T}$ denotes the transpose of the inverse of $\bar{\mathbf{C}}$. $\bar{\mathbf{C}}$, $\bar{\mathbf{K}}$ and $\bar{\mathbf{f}}$ are assembled using the mean values of ρc, k, T_∞, h, and Q. The initial condition for equation (23) is given by

$$\bar{\mathbf{u}}(t=0) = \bar{\mathbf{u}}_0 \tag{30}$$

$$\mathbf{V}_{\mathbf{u},\mathbf{u}} = \sigma^2_{T_0}\mathbf{I} \tag{31}$$

where $\mathbf{I}$ is an $n_{\mathrm{nod}} \times n_{\mathrm{nod}}$ unity matrix. Further, since the initial temperature is uncorrelated with k, ρc, h, T_∞, and Q, the other initial conditions are equal to null matrices of appropriate dimension. Equations (23)–(31) constitute the variance propagation algorithm for stochastic heat conduction problems.

Observe that the above algorithm can be extended to take into account nonlinear heat conduction with temperature dependent thermal properties since equations (21) and (22) are applicable to general nonlinear systems. The corresponding algorithm has a similar overall structure as the above algorithm and has been described in detail in ref. [12]. As it is essentially based on a linearization of the finite element formulation of the (nonlinear) heat conduction equation, it can however be

expected to be sufficiently accurate for smooth nonlinear heat conduction problems only. The applicability of this algorithm for heat conduction problems with phase changes is currently being investigated by the authors.

An algorithm which takes into account possible statistical correlations between random parameters has also been derived in ref. [12].

4. Implementation

Equation (24) is of the general form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}(t) = \mathbf{A}\mathbf{V}(t) + \mathbf{V}(t)\mathbf{A}^T + \mathbf{B}^{(t)}$$

with **V**, **A**, and **B** square matrices of equal dimension, and is called a Lyapunov matrix differential equation. Equations (25)–(27) are of the general form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}(t) = \mathbf{A}\mathbf{V}(t) + \mathbf{V}(t)\mathbf{B} + \mathbf{C}(t) \tag{32}$$

with **A**, **B** square matrices, and **V** and **C** matrices which are in general not square. Equation (32) is called a Sylvester matrix differential equation. Equations (28) and (29) are of the form

$$\bar{\mathbf{C}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}(t) + \bar{\mathbf{K}}\mathbf{V}(t) = \mathbf{h}(t)$$

with $\bar{\mathbf{C}}$, $\bar{\mathbf{K}}$ the finite element matrices, and **V** and **h** vectors of dimension n_{nod}. This structure is similar to that of equation (23), and further on it will be outlined that this fact can be exploited advantageously.

The matrices $\mathbf{V}_{\mathbf{x}_{T_\infty},\mathbf{x}_T}$, $\mathbf{V}_{\mathbf{x}_h,\mathbf{x}_h}$, and $\mathbf{V}_{\mathbf{x}_Q,\mathbf{x}_Q}$ in equations (25)–(27) can be computed by straightforward application of the variance propagation algorithm to the equations (10)–(12), respectively, which yields for example for T_∞

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}_{\mathbf{x}_{T_\infty},\mathbf{x}_{T_\infty}} = \mathbf{A}_{T_\infty}\mathbf{V}_{\mathbf{x}_{T_\infty},\mathbf{x}_{T_\infty}} + \mathbf{V}_{\mathbf{x}_{T_\infty},\mathbf{x}_{T_\infty}}\mathbf{A}_{T_\infty}^T + \mathbf{B}_{T_\infty}\sigma_{T_\infty}^2\mathbf{B}_{T_\infty}^T \tag{33}$$

It can be proven that AR(m) processes driven by stationary white noise are stationary [11]. This implies that the mean and the covariance of the AR(m) process does not change in time. Consequently, the time derivatives in the left hand sides of equation (33) vanish and the following algebraic matrix Lyapunov equation is obtained

$$\mathbf{A}_{T_\infty}\mathbf{V}_{\mathbf{x}_{T_\infty},\mathbf{x}_{T_\infty}} + \mathbf{V}_{\mathbf{x}_{T_\infty},\mathbf{x}_{T_\infty}}\mathbf{A}_{T_\infty}^T + \mathbf{B}_{T_\infty}\sigma_{T_\infty}^2\mathbf{B}_{T_\infty}^T = \mathbf{0} \tag{34}$$

Similarly,

$$\mathbf{A}_h\mathbf{V}_{x_h,x_h} + \mathbf{V}_{x_h,x_h}\mathbf{A}_h^T + \mathbf{B}_h + \sigma_h^2\mathbf{B}_h^T = 0 \tag{35}$$

$$\mathbf{A}_Q\mathbf{V}_{x_Q,x_Q} + \mathbf{V}_{x_Q,x_Q}\mathbf{A}_Q^T + \mathbf{B}_Q + \sigma_Q^2\mathbf{B}_Q^T = 0 \tag{36}$$

The above equations can now be combined conveniently in the algorithm outlined in Table 1.

The algorithm was programmed on top of the existing finite element code DOT [13]. Step 2 is calculated in advance, as well as the partial derivatives of **K**, **C** and **f** with respect to the random parameters. The latter can be assembled elementwise in the same way as **K**, **C** and **f** themselves [4]. The other steps are merged into a time stepping scheme in which the mean temperature vector and all covariance matrices are updated each time step. As in the original code, the linear differential systems (23), (28) and (29) are solved using an implicit Euler finite difference algorithm, e.g.,

$$\left(\frac{\mathbf{C}}{\Delta t} + \mathbf{K}\right)\mathbf{u}_{t+\Delta t} - \frac{\mathbf{C}}{\Delta t}\mathbf{u}_t = \mathbf{f}_{t+\Delta t} \tag{37}$$

Observe that the matrix $\bar{\mathbf{C}}/\Delta t + \bar{\mathbf{K}}$ is to be triangularized only once.

The Lyapunov and Sylvester differential systems are also discretized using an implicit Euler method, and the resulting algebraic Lyapunov and Sylvester algebraic systems are solved as described in ref. [14]. The algorithm involves the Schur decomposition of a matrix which also must be done only once, so that some additional computing time economization is possible. More details can be found in ref. [12].

Table 1
Variance propagation algorithm

Step 1	Computer $\bar{\mathbf{u}}$ from equation (23) with initial condition equation (30)
Step 2	Solve the Lyapunov matrix equations (34)–(36)
Step 3	Compute $\mathbf{V}_{\mathbf{u},\mathbf{x}_{T_\infty}}$, $\mathbf{V}_{\mathbf{u},\mathbf{x}_h}$, $\mathbf{V}_{b,x_Q}$, $\mathbf{V}_{\mathbf{u},k}$, and $\mathbf{V}_{\mathbf{u},\rho c}$ from equations (25)–(29)
Step 4	Compute $\mathbf{V}_{\mathbf{u},\mathbf{u}}(t)$ by solving the Lyapunov matrix differential equation (24)

5. Results and discussion

The variance propagation algorithm yields essentially a first order approximation of the mean value and covariance matrix of the temperature vector. In the test problems several parameters are considered to be random at the same time in order to (i) investigate whether still an acceptable accuracy and efficiency could be obtained when comparing Monte Carlo simulations with the variance propagation algorithm and (ii) to provide more realistic test cases. In ref. [12] a good agreement has been shown between the results obtained with the variance propagation algorithm and with the Monte Carlo method when each individual parameter was allowed to vary at a time.

The AR(m) process samples for the Monte Carlo simulations were generated by numerical solution of the governing differential equations with artificially generated

white noise sequences. Gaussian white noise was generated using NAG random generator G05DDF. All test problems were solved on a HP 9000/720 UNIX workstation.

5.1. Example 1. Parallel sided slab with heat generation

The first test problem consisted of a parallel sided metal slab with length $L = 10$ cm. The boundary at $x = 0$ cm is adiabatic, whereas the boundary at $x = L$ is kept at 0°C. The thermophysical properties are random variables with $k = 100$ W/m °C; $\sigma_k = 0.1 \times \bar{k}$; $\rho c = 7 \cdot 10^6$ J/m³ °C; $\sigma_{\rho c} = 0.1 \times \rho c$. Inside the slab heat is generated according to a AR(2) process (5)–(4) with $a_1 = 0.072$, $a_2 = 0.00072$, $\sigma_w^2 = 1.0368 \times 10^6$ and $\bar{Q} = 10^6$ J/m³. It can be calculated [12] that $\sigma_Q = 0.1 \times \bar{Q}$. The initial temperature is a random variable with $\bar{T}_0 = 0$°C, $\sigma_{T_0} = 1$°C.

For the finite element analysis the region $[0, L]$ was subdivided in 10 identical quadrilateral elements with four nodes per element (only two dimensional planar and axisymmetric elements are provided in the DOT finite element code) aligned parallel to the heat flow. All four edges of the finite elements were of equal size. The boundary of the computational grid at $x = 0$ and the boundaries parallel to x-axis were adiabatic, and the boundary at $x = L$ was kept at 0°C. Δt was set at 10 s for the transient analysis. Preliminary deterministic simulations showed that the global error of the finite element approximation with respect to the analytical solution was of the order of 0.1°C.

A sample of the random process heat generation and the corresponding temperature course at various positions inside the slab are shown in Fig. 1. As expected the magnitude of the fluctuations of the temperature inside the slab increase with increasing distance from the deterministic boundary at $x = L$. In Figs 2 and 3 the time course of the mean and the variance of the temperature is shown for three different positions in the slab. There is clearly a good agreement between the results obtained by means of the variance propagation algorithm aned the Monte Carlo method with 1000 runs. However, the variances obtained by means of the Monte Carlo method with 100 runs are scattered.

For comparison purposes the computational times were normalized with respect to the time to solve a deterministic problem (1.4 s). The variance propagation algorithm required 6.4 times units, whereas the Monte Carlo method required 128 (100 runs) and 1274 (1000 runs) time units.

5.2. Example 2. Beam with non-square cross section and internal heat generation

The second test problem consisted of a beam of infinite length with non-square cross section (Fig. 4). At the four sides of the geometry a convection boundary condition was applied. All random processes are of the type AR(1), with $a_1 = 0.01$. The thermophysical properties are random variables with $\bar{k} = 1.0$ W/m °C, $\rho c = 1.0 \times 10^6$ J/m³ °C, $\sigma_k = 0.1 \times \bar{k}$ and $\sigma_{\rho c} = 0.1 \times \rho c$. The heat generation is a random process with $\sigma_w^2 = 2 \times 10^6$, $\bar{Q} = 1 \cdot 10^5$ W/m³, and $\sigma_Q = 0.1 \times \bar{Q}$. For the boundary condition, $\bar{h} = 100$ W/m² °C, $\bar{T}_\infty = 0$°C, $\sigma_{w,h}^2 = \sigma_{w,T_\infty}^2 = 2.0$, $\sigma_h = 0.1 \times \bar{h}$ and $\sigma_{T_\infty} = 10$°C. The initial temperature is a random variable with $\bar{T}_0 = 0$°C, $\sigma_{T_0} = 10$°C.

The finite element grid consisted of 100 quadrilateral elements with four nodes/element, yielding a total of 121 nodes. The time step was equal to 60 s.

In Figures 5 and 6 the results obtained by means of the different methods are compared. As with the first test problem, there is a good agreement between the results obtained with the variance propagation algorithm and the Monte Carlo method with 1000 runs. The variances calculated by means of the Monte Carlo method with 100 runs differ considerably with those calculated by means of the other methods. For short times it can be observed that

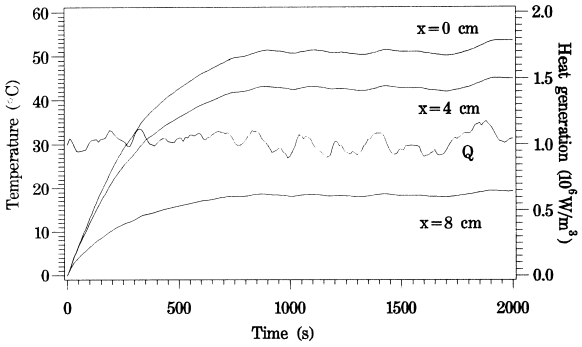

Fig. 1. Realization of stochastic heat generation and corresponding temperature at various positions in parallel sided slab.

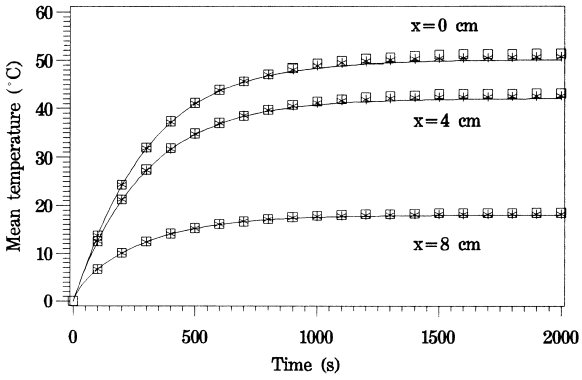

Fig. 2. Mean temperature at various positions in parallel sided slab with stochastic parameters; —: variance propagation algorithm; □: Monte Carlo ($n_{MC} = 100$); ★: Monte Carlo ($n_{MC} = 1000$).

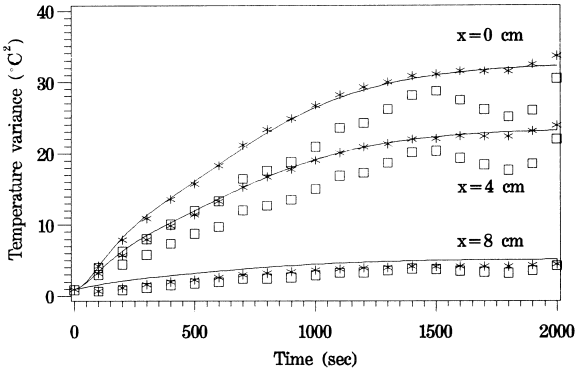

Fig. 3. Temperature variance at various positions in parallel sided slab with stochastic parameters; —: variance propagation algorithm; □: Monte Carlo ($n_{MC} = 100$); ★: Monte Carlo ($n_{MC} = 1000$).

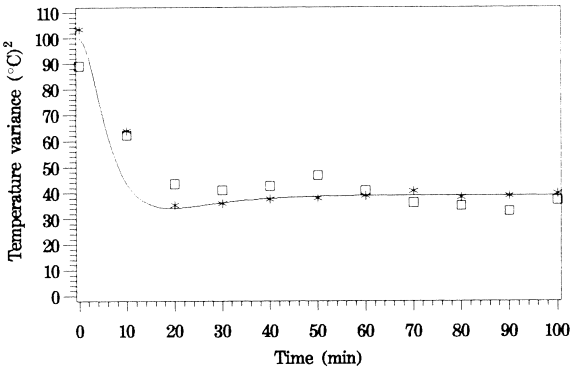

Fig. 6. Temperature variance at the position $(x, y) = (3$ cm, 5 cm) of the beam. —: variance propagation algorithm; □: Monte Carlo ($n_{MC} = 100$); ★: Monte Carlo ($n_{MC} = 1000$).

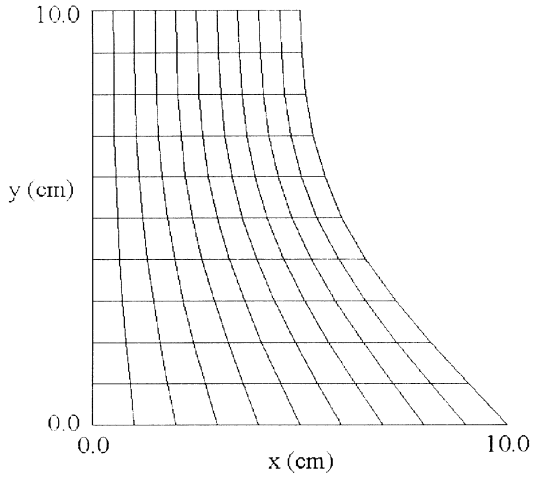

Fig. 4. Finite element grid of a beam with non-square cross section.

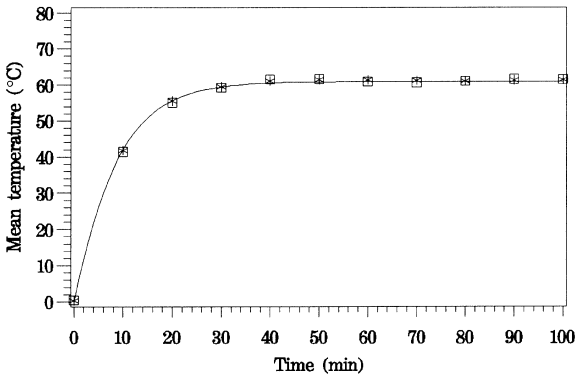

Fig. 5. Temperature variance at the position $(x, y) = (3$ cm, 5 cm) of the beam. —: variance propagation algorithm; □: Monte Carlo ($n_{MC} = 100$); ★: Monte Carlo ($n_{MC} = 1000$).

the agreement between the variance of the temperature obtained with the variance propagation algorithm and with the Monte Carlo method is not as good. This could be due to the fact that the variance propagation algorithm only provides a first order approximation of the mean value and covariance matrix of the temperature vector.

The normalized computational times were equal to 306, 1965 and 19 074 s, for the variance propagation and Monte Carlo (100 and 1000 runs), respectively.

6. Conclusions

In this article a variance propagation algorithm was derived for heat conduction problems with parameters involving stochastic fluctuations in time. The algorithm is based on the spatial discretization of the heat conduction equation by means of the finite element method, and involves the simultaneous solution of Sylvester and Lyapunov differential systems. It was implemented on top of an existing finite element code.

The performance of the variance propagation algorithm was compared with that of the direct Monte Carlo method. It was shown that at least 1000 Monte Carlo runs were required to obtain variances with an acceptable accuracy. In this case there was a good agreement with the results obtained by means of the variance propagation algorithm. However, the latter was up to six and 60 times faster than the Monte Carlo method with 100 and 1000 runs, respectively.

Acknowledgements

The authors wish to thank the European Commission (FAIR project CT96-1192) and the Flemish Minister for Science and Technology for financial support. Bart Nicolaï

is a Research Fellow with the Flanders Fund for Scientific Research (F.W.O. Vlaanderen).

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